A centennial of the Zaremba–Hopf–Oleinik Lemma

A.I. Nazarov[*],
Saint-Petersburg University,
e-mail: al.il.nazarov@gmail.com

We consider the Hopf–Oleinik normal derivative lemma for elliptic and parabolic equations under minimal restrictions on lower-order coefficients. Boundary gradient estimates for solutions are also established.

1 Introduction

Qualitative theory of partial differential equations is in intensive development over last half of century. In this paper we discuss the Hopf–Oleinik Lemma, one of the most important tools in studying solutions to elliptic and parabolic equations, in particular, the key argument in the proof of uniqueness theorems.

For the Laplace operator this property is well known for one hundred years, starting from a pioneer paper of Zaremba [Z], and reads as follows. Let $\partial\Omega \in \mathcal{C}^2$ and let $\mathcal{L} = -\Delta$. Then, if $0 \in \partial\Omega$, we have

$$\mathcal{L}u = f \geq 0 \ \text{ in } \ \Omega; \quad u(x) > u(0) \ \text{ in } \ \Omega \quad \Longrightarrow \quad \frac{\partial u}{\partial \mathbf{n}}(0) < 0. \tag{\textbf{ZHO}}$$

For general operators of non-divergence type with bounded measurable coefficients this result was established in elliptic case independently by E. Hopf [Ho] and O.A. Oleinik [O] and in parabolic case by L. Nirenberg [Ni]. Later the efforts of many mathematicians were aimed at the reduction of the boundary smoothness[1]. They established that the sharp condition for (**ZHO**) to fulfil is the Dini condition for the boundary normal, see, e.g., [Hi]. In a weakened form (the existence a boundary point x^0 in any neighborhood of the origin and a direction ℓ such that $\frac{\partial u}{\partial \ell}(x^0) < 0$) this fact holds true for a much wider class of domains including all Lipschitz ones, see [Na] for elliptic equations and [K] for parabolic ones. Note that all these results are related to classical solutions, i.e. $u \in \mathcal{C}^2(\Omega)$ in elliptic case and $u \in \mathcal{C}^{2,1}(Q)$ in parabolic case.

Now let us consider generalized (*strong*) solutions for non-divergence type equations

$$\mathcal{L}u \equiv -a_{ij}(x)D_iD_ju + b_i(x)D_iu = f(x); \tag{\textbf{NDE}}$$

$$\mathcal{M}u \equiv \partial_t u - a_{ij}(x;t)D_iD_ju + b_i(x;t)D_iu = f(x;t), \tag{\textbf{NDP}}$$

i.e. we assume $D(Du) \in L_{n,loc}(\Omega)$ in (**NDE**) and $\partial_t u$, $D(Du) \in L_{n+1,loc}(Q)$ in (**NDP**) (in the parabolic case also some anisotropic spaces are admissible).

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[1]See also an earlier paper [G] for equations with Hölder continuous leading coefficients.

We always suppose that operators under consideration are uniformly elliptic (parabolic), i.e. for all values of arguments

$$\nu|\xi|^2 \le a_{ij}(\cdot)\xi_i\xi_j \le \nu^{-1}|\xi|^2, \qquad \xi \in \mathbb{R}^n, \tag{1}$$

where ν is a positive constant. Note that we can also assume $a_{ij} \equiv a_{ji}$ without loss of generality.

The properties of generalized solutions to the equations (**NDE**)-(**NDP**), under assumption that leading coefficients a_{ij} are only measurable, were investigated in a number of papers[2]. The problem of our interest is how "bad" may be lower-order coefficients b_i to ensure the Hopf–Oleinik Lemma to hold true. We provide sharp conditions for this. We also touch the topic closely related to (**ZHO**), especially in idea of proof, namely, the gradient estimates at the boundary.

Note that for divergence type equations

$$-D_i\big(a_{ij}(x)D_ju\big) + b_i(x)D_iu = 0; \tag{\textbf{DE}}$$

$$\partial_t u - D_i\big(a_{ij}(x;t)D_ju\big) + b_i(x;t)D_iu = 0. \tag{\textbf{DP}}$$

(**ZHO**) does not hold. The simplest counterexample is the function $u(x) = x_2^2 + 2x_2|x_1|$ which is positive in the upper half-plane, satisfies the equation (**DE**) with

$$(a_{ij}) = \begin{bmatrix} 1 & -\mathrm{sign}(x_1) \\ -\mathrm{sign}(x_1) & 2 \end{bmatrix}; \qquad b_i \equiv 0; \qquad f \equiv 0,$$

but $u(0,0) = 0$ and $D_2u(0,0) = 0$.

Moreover, even continuity of a_{ij} does not improve the situation. Let us describe corresponding counterexample[3].

Let Ω be a convex domain, and let $0 \in \partial\Omega$. Assume that at the neighborhood of the origin $\partial\Omega$ is the graph of a function $x_n = \phi(x')$. Finally, suppose that $\phi \in \mathcal{C}^1$ but $D'\phi$ is not Dini continuous at the origin.

As it was mentioned the Hopf–Oleinik lemma for the Laplacian fails in such domain. Now we rectify the boundary near the origin and obtain an operator of the form (**DE**) with *continuous* leading coefficients and $b_i \equiv 0$ for which (**ZHO**) fails in smooth domain. Considering functions depending only on spatial variables we see that this example works also for the parabolic operator (**DP**).

The paper is organized as follows. In Section 2 we deal with elliptic equations, Section 3 is devoted to parabolic equations. In both sections we use the "composite" variant of the A.D. Aleksandrov maximum estimate ([Li00]; see also [AN95] for a weaker version) and slightly modify classical techniques due to Ladyzhenskaya–Ural'tseva [LU88], see also [S10].

Let us recall some notation. $x = (x_1, \dots, x_{n-1}, x_n) = (x', x_n)$ is a vector in $\mathbb{R}^n$, $n \ge 2$, with the Euclidean norm $|x|$; $(x;t)$ is a point in $\mathbb{R}^{n+1}$.

Ω is a domain in $\mathbb{R}^n$ and $\partial\Omega$ is its boundary; $\mathbf{n} = (\mathbf{n}_i(x))$ is the unit vector of the outward normal to $\partial\Omega$ at the point x.

For a cylinder $Q = \Omega\times]0,T[$ we denote by $\partial''Q = \partial\Omega\times]0,T[$ its lateral surface and by $\partial'Q = \partial''Q \cup \{\overline{\Omega} \times \{0\}\}$ its parabolic boundary.

[2]We mention in this connection a quite recent paper [A-Z] discussing some degenerate elliptic equations.

[3]See also [GT, Problem 3.9].

We define

$$\begin{array}{ll} B_r(x^0) = \{x \in \mathbb{R}^n : |x - x^0| < r\}, & B_r = B_r(0); \\ \mathcal{B}_{r,h}(x^{0\prime}) = \{x \in \mathbb{R}^n : |x' - x^{0\prime}| < r, 0 < x_n < h\}; & \mathcal{B}_{r,h} = \mathcal{B}_{r,h}(0); \\ Q_r(x^0; t^0) = B_r(x^0) \times]t^0 - r^2; t^0[, & Q_r = Q_r(0;0); \\ \mathcal{Q}_{r,h}(x^{0\prime}; t^0) = \mathcal{B}_{r,h}(x^{0\prime}) \times]t^0 - r^2; t^0[, & \mathcal{Q}_{r,h} = \mathcal{Q}_{r,h}(0;0). \end{array}$$

The indices i, j vary from 1 to n. Repeated indices indicate summation.

The symbol D_i denotes the operator of differentiation with respect to x_i; in particular, $Du = (D_1u, \dots, D_nu) = (D'u, D_nu)$ is the gradient of u. $\partial_t u$ stands for the derivative of u with respect to t.

We denote by $\|\cdot\|_{q,\Omega}$ the norm in $L_q(\Omega)$. We introduce two scales of anisotropic spaces:

$L_{q,\ell}(Q) = L_\ell\big(]0,T[\to L_q(\Omega)\big)$ with the norm $\|f\|_{q,\ell,Q} = \big\|\|f(\cdot;t)\|_{q,\Omega}\big\|_{\ell,]0,T[}$;

$\widetilde{L}_{q,\ell}(Q) = L_q\big(\Omega \to L_\ell(]0,T[)\big)$ with the norm $\|f\|^{\sim}_{q,\ell,Q} = \big\|\|f(x;\cdot)\|_{\ell,]0,T[}\big\|_{q,\Omega}$.

Obviously, $L_{q,q}(Q) = \widetilde{L}_{q,q}(Q) = L_q(Q)$. Further, by the Minkowskii inequality,

$$\|f\|^{\sim}_{q,\ell,Q} \le \|f\|_{q,\ell,Q} \quad \text{for} \quad q \ge \ell; \qquad \|f\|_{q,\ell,Q} \le \|f\|^{\sim}_{q,\ell,Q} \quad \text{for} \quad q \le \ell.$$

We denote by $\widehat{L}_{q,\ell}(Q)$ the space

$$L_{q,\ell}(Q) \cap \widetilde{L}_{q,\ell}(Q) = \begin{cases} L_{q,\ell}(Q), & q \ge \ell; \\ \widetilde{L}_{q,\ell}(Q), & q \le \ell \end{cases}$$

with the norm $|\!|\!|f|\!|\!|_{q,\ell,Q} = \max\{\|f\|_{q,\ell,Q}, \|f\|^{\sim}_{q,\ell,Q}\}$.

Remark 1. Note that we always deal with the space $\widehat{L}_{q,\ell}(Q)$ i.e. take the more strong of two norms. The reason is that up to now anisotropic versions of the Aleksandrov–Krylov maximum principle (see [N87], [N01]) are proved only in terms of stronger norm.

We set $f_+ = \max\{f, 0\}$, $\quad f_- = \max\{-f, 0\}$.

Following [Li00], we say that $\omega : \ [0,1] \to \mathbb{R}_+$ belongs to the class $\mathcal{D}_1$ if $\omega(1) = 1$, ω is continuous and increasing while $\omega(\sigma)/\sigma$ is summable and decreasing. In this case we define $\mathcal{I}_\omega(s) = \int_0^s \frac{\omega(\sigma)}{\sigma}\, d\sigma$.

We use letters M, N, C (with or without indices) to denote various constants. To indicate that, say, N depends on some parameters, we list them in the parentheses: $N(\dots)$.

2 Elliptic case

Recall that in this section we assume $D(Du) \in L_{n,loc}(\Omega)$.

The next statement is a particular case of [Li00, Theorem 3.2].

Proposition 2.1. *Let $\mathcal{L}$ be an operator of the form (**NDE**) in a bounded, strictly Lipschitz domain Ω, and let the condition (1) be satisfied. Suppose also that the vector function $\mathbf{b} = (b_i)$ can be written as follows:*

$$\mathbf{b} = \mathbf{b}^{(1)} + \mathbf{b}^{(2)}; \qquad |\mathbf{b}^{(1)}| \in L_n(\Omega); \tag{2}$$

$$|\mathbf{b}^{(2)}| \le \mathfrak{B}\, \frac{\omega(d/\mathrm{diam}(\Omega))}{d}, \qquad \omega \in \mathcal{D}_1,$$

where $d = d(x) = \mathrm{dist}(x, \partial\Omega)$.

Then for any solution of $\mathcal{L}u = f$ in Ω satisfying $u\big|_{\partial\Omega} \le 0$, the following estimate holds:

$$u \le N_0 \cdot \frac{\operatorname{diam}(\Omega)}{\nu} \cdot \|f_+\|_{n,\{u>0\}},$$

provided $\|\mathbf{b}^{(1)}\|_{n,\Omega} \le \mathfrak{B}_0$, where N_0 and $\mathfrak{B}_0$ depend only on n, ν, $\mathfrak{B}$ and the Lipschitz constant of $\partial\Omega$.

Now we prove a quantitative version of tne maximum principle, the so-called "boundary growth lemma" (its versions for $|\mathbf{b}| \in L_n$ are proved, e.g., in [LU85, Lemma 2.5'] and [S10, Lemma 2.6]).

Lemma 2.2. *Let $\mathcal{L}$ be an operator of the form (**NDE**) in $\mathcal{B}_{\rho,\rho}$, $\rho \le R$, and let the conditions (1), (2) and*

$$|\mathbf{b}^{(2)}(x)| \le \mathfrak{B}\,\frac{\omega(x_n/R)}{x_n}, \qquad \omega \in \mathcal{D}_1, \tag{3}$$

be satisfied. Suppose also that $\|\mathbf{b}^{(1)}\|_{n,\mathcal{B}_{\rho,\rho}} \le \mathfrak{B}_0$ where $\mathfrak{B}_0 = \mathfrak{B}_0(n,\nu,\mathfrak{B})$ is the constant from Proposition 2.1. If u is a nonnegative solution of $\mathcal{L}u = f \ge 0$ in $\mathcal{B}_{\rho,\rho}$ satisfying $u \ge k$ on $\partial\mathcal{B}_{\rho,\rho} \cap \{x_n = 0\}$ for some $k > 0$ then for $\xi \le \frac{1}{2}$ the inequality

$$u \ge k \cdot \big(\beta(n,\nu,\mathfrak{B},\omega,\xi) - N_1(n,\nu,\mathfrak{B},\omega,\xi) \cdot (\|\mathbf{b}^{(1)}\|_{n,\mathcal{B}_{\rho,\rho}} + \mathfrak{B}\omega(\rho/R))\big) \tag{4}$$

holds in $\mathcal{B}_{(1-\xi)\rho,(1-\xi)\rho}$ with some positive constants β and N_1.

Proof. Consider the barrier function

$$w(x) = \Big(1 - A\,\frac{x_n}{\rho}\Big)^2 + 2(1+A)\,\big(\varphi(x_n/\rho) - \varphi(1/A)\big) - \frac{|x'|^2}{\rho^2}, \tag{5}$$

where (cf. [Li00])

$$\varphi(s) = \int\limits_0^s \Big(\exp\Big(\frac{\mathfrak{B}}{\nu}\,\mathcal{I}_\omega(\sigma)\Big) - 1\Big)\,d\sigma,$$

while $A \ge \frac{\sqrt{n-1}}{\nu}$ is a constant to be defined later.

Direct calculation shows that for $x \in \mathcal{B}_{\rho,\frac{\rho}{A}}$

$$-a_{ij}D_iD_jw \le \frac{2}{\rho^2}\cdot\big[-\nu A^2 - \nu(1+A)\varphi''(\tau) + (n-1)\nu^{-1}\big] \le -\frac{2\nu}{\rho^2}\,(1+A)\varphi''(\tau);$$

$$b_i^{(1)}D_iw \le |\mathbf{b}^{(1)}|\cdot\frac{2}{\rho}\cdot\big[A\big(1 - A\,\tau\big) + (1+A)\varphi'(\tau) + 1\big] \le |\mathbf{b}^{(1)}|\cdot\frac{2}{\rho}\,(1+A)(1+\varphi'(\tau));$$

$$b_i^{(2)}D_iw \le \frac{2}{\rho^2}\cdot\big[A\big(1 - A\,\tau\big) + (1+A)\varphi'(\tau) + 1\big]\cdot\mathfrak{B}\,\frac{\omega(\tau)}{\tau} \le \frac{2\mathfrak{B}}{\rho^2}\,(1+A)\,\frac{\omega(\tau)}{\tau}\,(1+\varphi'(\tau))$$

(here $\tau = x_n/R$). Since $\varphi''(\tau) = \frac{\mathfrak{B}}{\nu}\,\frac{\omega(\tau)}{\tau}\,(1+\varphi'(\tau))$, we have $\mathcal{L}w \le C_1(A,\nu,\mathfrak{B})|\mathbf{b}^{(1)}|\rho^{-1}$.

Further, $w(x) \le 0$ for $|x'| = \rho$, $0 < x_n < \frac{\rho}{A}$ and for $|x'| \le \rho$, $x_n = \frac{\rho}{A}$. Finally, $w(x) \le 1$ for $|x'| \le \rho$, $x_n = 0$. This gives $kw - u \le 0$ on $\partial\mathcal{B}_{\rho,\frac{\rho}{A}}$.

Proposition 2.1 gives for $x \in \mathcal{B}_{\rho,\frac{\rho}{A}}$

$$u(x) \ge kw(x) - kC_2\rho\cdot\|(\mathcal{L}w)_+\|_{n,\mathcal{B}_{\rho,\frac{\rho}{A}}} \ge kw(x) - kC_3\|(\mathbf{b}^{(1)}\|_{n,\mathcal{B}_{\rho,\frac{\rho}{A}}},$$

where C_2 depends only on n, ν and $\mathfrak{B}$ while C_3 depends on the same quantities and on A.

Now we observe that $\varphi(s) = o(s)$ as $s \to 0$. Thus, we can choose $A = A(n, \nu, \mathfrak{B}, \omega, \xi) \geq \frac{\sqrt{n-1}}{\nu}$ so large that $2(1+A)\varphi(1/A) \leq \frac{\xi}{2}$. Since $(1-\frac{\xi}{2})^2 - \frac{\xi}{2} - (1-\xi)^2 = \xi(1-\frac{3\xi}{4}) \geq \frac{5\xi}{8}$, this gives

$$u \geq k \cdot \Big(\frac{5\xi}{8} - C_4(n, \nu, \mathfrak{B}, \omega, \xi)\|\mathbf{b}^{(1)}\|_{n,\mathcal{B}_{\rho,\rho}}\Big)_+ \equiv k_1 \qquad \text{in} \quad \mathcal{B}_{(1-\xi)\rho, \frac{\xi\rho}{2A}}. \tag{6}$$

Now we consider the set $K_\rho = \mathcal{B}_{\rho,\rho} \setminus \mathcal{B}_{\rho,\frac{\xi\rho}{4A}}$. Note that coefficients $b_i^{(2)}$ are bounded on this set, and

$$\|\mathbf{b}^{(2)}\|_{n,K_\rho} \leq C_5(n)\mathfrak{B}\Big(\int\limits_{\frac{\xi}{4A}}^{1} \Big(\frac{\omega(s\rho/R)}{s}\Big)^n ds\Big)^{\frac{1}{n}} \leq C_6(n, \nu, \mathfrak{B}, \omega, \xi)\mathfrak{B}\omega(\rho/R). \tag{7}$$

We apply "the ink-spot expansion lemma" ([LU85, Lemma 2.2]) and obtain

$$u \geq k_1 \cdot \Big(\varkappa(n, \nu, \mathfrak{B}, \omega, \xi) - C_7(n, \nu, \mathfrak{B}, \omega, \xi)\|\mathbf{b}^{(1)} + \mathbf{b}^{(2)}\|_{n,K_\rho}\Big) \qquad \text{in} \quad \mathcal{B}_{(1-\xi)\rho,(1-\xi)\rho} \setminus \mathcal{B}_{(1-\xi)\rho,\frac{\xi\rho}{2A}}.$$

By (6) and (7) we arrive at (4). $\square$

Lemma 2.2'. *Let $\mathcal{L}$ be as in Lemma 2.2. If u is a nonnegative solution of $\mathcal{L}u = f \geq 0$ in $\mathcal{B}_{\rho,\rho}$ satisfying $u \geq k$ on $\partial\mathcal{B}_{\rho,\rho} \cap \{x_n = \rho\}$ for some $k > 0$ then for $\xi \leq \frac{1}{2}$ the inequality (4) holds in $\mathcal{B}_{(1-\xi)\rho,\rho} \setminus \mathcal{B}_{(1-\xi)\rho,\xi\rho}$.*

Proof. This statement is more simple than Lemma 2.2. Consider the set $\widetilde{K}_\rho = \mathcal{B}_{\rho,\rho} \setminus \mathcal{B}_{\rho,\frac{\xi\rho}{2}}$. Since coefficients $b_i^{(2)}$ are bounded on this set and $\|\mathbf{b}^{(2)}\|_{n,\widetilde{K}_\rho}$ is under control, we can apply standard boundary growth lemma, and the statement follows. $\square$

Remark 2. If we replace the assumption $f \geq 0$ by $f_- \in L_n(\mathcal{B}_{\rho,\rho})$, the estimate (4) holds true with additional term $-N_2(n, \nu, \mathfrak{B}, \omega, \xi)\rho \cdot \|f_-\|_{n,\mathcal{B}_{\rho,\rho}}$ in the right-hand side. The proof runs without changes.

Now we prove the main result of this Section.

Theorem 2.3. *Let $\mathcal{L}$ be an operator of the form (**NDE**) in $\mathcal{B}_{R,R}$, and let the conditions (1), (2) and (3) be satisfied. Suppose also that for $\rho \leq R$*

$$\|b_n^{(1)}\|_{n,\mathcal{B}_{\rho,\rho}} \leq \mathfrak{B}_1\omega(\rho/R). \tag{8}$$

Then

1. *Any solution of $\mathcal{L}u = f \leq 0$ in $\mathcal{B}_{R,R}$ such that $u|_{x_n=0} \leq 0$ and $u(0) = 0$ satisfies*

$$\sup_{0<x_n<R/2} \frac{u(0,x_n)}{x_n} \leq \frac{N_3^+}{R} \cdot \sup_{\mathcal{B}_{R/2,R/2}} u,$$

Consequently, if $D_nu(0)$ exists then $(D_nu)_+(0)$ is finite.

2. *Any positive solution of $\mathcal{L}u = f \geq 0$ in $\mathcal{B}_{R,R}$ such that $u(0) = 0$ satisfies $\inf\limits_{0<x_n<R/2} \frac{u(0,x_n)}{x_n} > 0$. Consequently, if $D_nu(0)$ exists, it is positive. If, in addition, $f \equiv 0$, the following estimate holds:*

$$\inf_{0<x_n<R/2} \frac{u(0,x_n)}{x_n} \geq N_3^- \cdot \frac{u(0,R/2)}{R/2}.$$

The constants $N_3^\pm$ depend on n, ν, $\mathfrak{B}$, $\mathfrak{B}_1$, ω and the the moduli of continuity of $|\mathbf{b}|$ in $L_n(\mathcal{B}_{R,R})$.

Proof. We introduce the sequence of cylinders $\mathcal{B}_{\rho_k,h_k}$, $k\ge 0$, where $\rho_k=2^{-k}\rho_0$, $h_k=\zeta_k\rho_k$, while $\rho_0\le R$ and the sequence $\zeta_k\downarrow 0$ will be chosen later.

Denote by $M_k^\pm$, $k\ge 1$, the quantities

$$M_k^+=\sup_{\mathcal{B}_{\rho_k,h_{k-1}}}\frac{u(x)}{\max\{x_n,h_k\}}\ge\sup_{\mathcal{B}_{\rho_k,h_{k-1}}\setminus\mathcal{B}_{\rho_k,h_k}}\frac{u(x)}{x_n};\qquad M_k^-=\inf_{\mathcal{B}_{\rho_k,h_{k-1}}\setminus\mathcal{B}_{\rho_k,h_k}}\frac{u(x)}{x_n}.$$

Note that in the case **2** $M_k^->0$.

We define two function sequences

$$v_k^1=u-M_k^+h_k\,\frac{\varphi^+(x_n/R)}{\varphi^+(h_k/R)};\qquad v_k^2=M_k^-h_k\,\frac{\varphi^-(x_n/R)}{\varphi^-(h_k/R)}-u,$$

where, similarly to Lemma 2.2,

$$\varphi^\pm(s)=\int\limits_0^s\exp\Big(\mp\frac{\mathfrak{B}}{\nu}\,\mathcal{I}_\omega(\sigma)\Big)d\sigma,\tag{9}$$

and denote

$$\begin{array}{llll} V_k=v_k^1; & M_k=M_k^+; & \Phi=\varphi^+ & \text{in the case } \mathbf{1};\\ V_k=v_k^2; & M_k=M_k^-; & \Phi=\varphi^- & \text{in the case } \mathbf{2}. \end{array}$$

It is easy to see that $V_k\big|_{x_n=0}\le 0$ while the definition of M_k gives $V_k\le 0$ on the top of the cylinder $\mathcal{B}_{\rho_k,h_k}$.

To estimate V_k, we refine a trick from [S10]. Let $x^0\in\mathcal{B}_{\rho_k-h_k,h_k}$. Assume first that $x_n^0\le\frac{h_k}{2}$. Then we apply Lemma 2.2[4] to the (positive) function $M_kh_k-V_k$ in $\mathcal{B}_{h_k,h_k}(x^{0\prime})$ (with regard to Remark 2). This gives for $x\in\mathcal{B}_{\frac{h_k}{2},\frac{h_k}{2}}(x^{0\prime})$

$$\begin{aligned} M_kh_k-V_k(x)\ge M_kh_k\cdot\big[\beta(n,\nu,\mathfrak{B},\omega,1/2)-&\\ -N_1(n,\nu,\mathfrak{B},\omega,1/2)\cdot(\|\mathbf{b}^{(1)}\|_{n,\mathcal{B}_{\rho_k,\rho_k}}+\mathfrak{B}\omega(\rho_k/R))\big]-&\\ -N_2(n,\nu,\mathfrak{B},1/2)h_k\cdot\|(\mathcal{L}V_k)_+\|_{n,\mathcal{B}_{h_k,h_k}(x^{0\prime})}.& \end{aligned}\tag{10}$$

We suppose that ρ_0/R is so small that the quantity in the square brackets is greater that $\frac{\beta}{2}$. Further, direct calculation similar to Lemma 2.2 shows that the assumptions of theorem imply

$$\mathcal{L}V_k\le M_k|b_n^{(1)}|\,\Phi'(x_n/R)\,\frac{h_k/R}{\Phi(h_k/R)}\qquad\text{in}\quad\mathcal{B}_{\rho_k,h_k}.$$

Note that φ^+ is concave, φ^- is convex, and both of them are increasing. Therefore,

$$\Phi'(x_n/R)\,\frac{h_k/R}{\Phi(h_k/R)}\le\frac{\max\{1,\Phi'(1)\}}{\Phi(1)}.$$

Substituting these inequalities into (10) we arrive at

$$V_k(x)\le M_kh_k\cdot\Big[1-\beta/2+C_8(n,\nu,\mathfrak{B},\omega)\|b_n^{(1)}\|_{n,\mathcal{B}_{h_k,h_k}(x^{0\prime})}\Big]\qquad\text{for}\quad x\in\mathcal{B}_{\frac{h_k}{2},\frac{h_k}{2}}(x^{0\prime}).$$

[4] To proceed we suppose that ρ_0/R is so small that $\|\mathbf{b}\|_{n,\mathcal{B}_{\rho_0,\rho_0}}\le\mathfrak{B}_0$.

In particular, this estimate is valid for $x = x^0$. If $x^0_n \ge \frac{h_k}{2}$, we get the same estimate using Lemma 2.2' instead of Lemma 2.2.

Taking supremum w.r.t x^0, we obtain

$$\sup_{\mathcal{B}_{\rho_k - h_k, h_k}} V_k \le M_k h_k \cdot \big(1 - \beta/2 + C_8 \mathfrak{B}_1 \omega(\rho_k/R)\big).$$

Repeating previous arguments provides for $m \le \frac{\rho_k}{h_k}$

$$\sup_{\mathcal{B}_{\rho_k - m h_k, h_k}} V_k \le M_k h_k \cdot \Big((1 - \beta/2)^m + C_8 \mathfrak{B}_1 \frac{\omega(\rho_k/R)}{\beta/2}\Big).$$

Setting $m = \lfloor \frac{\rho_{k+1}}{h_k} \rfloor$, we arrive at

$$\sup_{\mathcal{B}_{\rho_{k+1}, h_k}} V_k \le \frac{M_k h_k}{1 - \beta/2} \cdot \Big(\exp\Big(-\lambda \frac{\rho_{k+1}}{h_k}\Big) + C_8 \mathfrak{B}_1 \frac{\omega(\rho_k/R)}{\beta/2}\Big),$$

where $\lambda = -\ln(1 - \beta/2) > 0$.

Therefore, for $x \in \mathcal{B}_{\rho_{k+1}, h_k}$

$$\frac{V_k(x)}{\max\{x_n, h_{k+1}\}} \le M_k \gamma_k, \tag{11}$$

where $\gamma_k = \frac{1}{1-\beta/2} \frac{\zeta_k}{2\zeta_{k+1}} \cdot \big(\exp\big(-\frac{\lambda}{2\zeta_k}\big) + C_8 \mathfrak{B}_1 \frac{\omega(\rho_k/R)}{\beta/2}\big)$.

Estimate (11) implies in the cases **1** and **2**, respectively,

$$M^{\pm}_{k+1} \lesseqgtr M^{\pm}_k (\delta^{\pm}_k \pm \gamma_k), \tag{12}$$

where

$$\begin{aligned}
\delta^+_k &= \frac{h_k}{\varphi^+(h_k/R)} \cdot \sup_{0 \le x_n \le h_k} \frac{\varphi^+(x_n/R)}{\max\{x_n, h_{k+1}\}} = \frac{h_k}{\varphi^+(h_k/R)} \cdot \frac{\varphi^+(h_{k+1}/R)}{h_{k+1}}; \\
\delta^-_k &= \frac{h_k}{\varphi^-(h_k/R)} \cdot \inf_{h_{k+1} \le x_n \le h_k} \frac{\varphi^-(x_n/R)}{x_n} = \frac{h_k}{\varphi^-(h_k/R)} \cdot \frac{\varphi^-(h_{k+1}/R)}{h_{k+1}}.
\end{aligned} \tag{13}$$

Since $\lim\limits_{s \to 0+} \frac{\Phi(s)}{s} = 1$, we have

$$\prod_k \delta^{\pm}_k = \frac{h_1/R}{\Phi(h_1/R)} \lesseqgtr \frac{1}{\Phi(1)}.$$

Thus, (12) gives in the cases **1** and **2**, respectively,

$$M^{\pm}_{k+1} \lesseqgtr \frac{M^{\pm}_1}{\Phi(1)} \cdot \prod_{j=1}^{k} \big(1 \pm \frac{\gamma_j}{\delta^{\pm}_j}\big) \lesseqgtr \frac{M^{\pm}_1}{\Phi(1)} \cdot \prod_{j=1}^{k} (1 \pm \gamma_j \cdot \max\{1, \Phi(1)\}).$$

We set $\zeta_k = \frac{1}{k+k_0}$ and choose k_0 so large and ρ_0/R so small that $\gamma_1 \cdot \max\{1, \Phi(1)\} \le \frac{1}{2}$. Note that k_0 and ρ_0/R satisfying all the conditions imposed depend only on n, ν, ω, $\mathfrak{B}$ and $\mathfrak{B}_1$.

Now we observe that the first terms in γ_k form a convergent series. The same is true for the second terms, since

$$\sum_{k=1}^{\infty} \omega\big(2^{-k} \rho_0/R\big) \asymp \int_0^{\infty} \omega\big(2^{-s} \rho_0/R\big)\, ds \asymp \mathcal{I}_\omega(\rho_0/R).$$

Thus, the series $\sum_k \gamma_k$ converges. Therefore, the infinite products $\Pi^\pm = \prod\limits_k (1 \pm \gamma_k \cdot \max\{1, \Phi(1)\})$ also converge, and we obtain in the cases **1** and **2**, respectively,

$$M_k^\pm \lesseqgtr \frac{\Pi^\pm M_1^\pm}{\Phi(1)}, \qquad k > 1.$$

Thus, all M_k^+ are bounded in the case **1**, and all M_k^- are separated from zero in the case **2**.

Further, we note that $M_1^+ \le \frac{1}{h_1} \sup\limits_{\mathcal{B}_{R/2,R/2}} u$. This completes the proof of the statement **1**.

If $f \equiv 0$, then we set $K = \mathcal{B}_{R,R} \setminus \mathcal{B}_{R,h_1/2}$. Similarly to Lemma 2.2, $\|\mathbf{b}^{(1)} + \mathbf{b}^{(2)}\|_{n,K}$ is bounded. Therefore, by the Harnack inequality ([S10, Theorem 3.3]), $M_1^- \asymp \frac{u(0,R/2)}{R/2}$. This completes the proof of the statement **2**. $\square$

Remark 3. If we replace in the case **1** the assumption $f \le 0$ by $f = f^{(1)} + f^{(2)}$ with

$$\|f_+^{(1)}\|_{n,\mathcal{B}_{\rho,\rho}} \le \mathfrak{F}_1 \omega(\rho/R), \qquad f_+^{(2)} \le \mathfrak{F}_2 \frac{\omega(x_n/R)}{x_n},$$

the estimate

$$\sup_{0<x_n<R/2} \frac{u(0,x_n)}{x_n} \le N_3^+ \cdot \Big(\frac{1}{R} \sup_{\mathcal{B}_{R/2,R/2}} u + \mathfrak{F}_1 + \mathfrak{F}_2\Big)$$

remains valid. The proof runs with minor changes.

Let us compare Theorem 2.3 with results known earlier. Surely, the proof of (**ZHO**) for classical solutions works also for strong solutions if we apply the Aleksandrov maximum principle ([Al]; see also a survey [N05], where the history of this topic is presented). So, it was known long ago for $b_i \in L_\infty$.

In [Li85] the Hopf–Oleinik Lemma was proved for classical solutions of (**NDE**) in $\mathcal{C}^{1+\mathcal{D}}$ domains. Note that in this case one can locally rectify $\partial\Omega$ using the regularized distance ([Li85, Theorem 2.1]). After this Theorem 4.1 [Li85] follows from a particular case $\mathbf{b}^{(1)} \equiv 0$ of Theorem 2.3, part **2**. Similarly, the boundary gradient estimates obtained in [Li86] can be reduced to the same particular case of Theorem 2.3, part **1**.

The boundary gradient estimates for solution to (**NDE**) were established in [LU88] provided $\mathbf{b} \in L_q$, $q > n$; the Hopf–Oleinik Lemma under the same condition was announced in [NU]. In [S10] the second part of Theorem 2.3 is proved for (**NDE**) under assumption $\mathbf{b} \in L_n$, $b_n \in L_q$, $q > n$. In [AN95] the first part of Theorem 2.3 was proved for composite coefficients with $\omega(\sigma) = \sigma^\alpha$, $\alpha \in]0,1[$.

To compare our result with [S08], we need an auxiliary statement.

Lemma 2.4. *Let $\Psi : [0,\sigma_0] \to \mathbb{R}_+$ be a nondecreasing function. Then there exist nondecreasing $\mathcal{C}^1$ functions $\Psi^\pm : [0,\sigma_0] \to \mathbb{R}_+$ such that $\Psi^- \le \Psi \le \Psi^+$, and*

1) if $\Psi(\sigma)/\sigma^2$ is summable then $(\Psi^+)'(\sigma)/\sigma$ is summable;

2) if $\Psi(\sigma)/\sigma^2$ is nonsummable then $(\Psi^-)'(\sigma)/\sigma$ is nonsummable.

Proof. Without loss of generality we can assume $\sigma_0 = 1$. Consider the function $\Psi_1(\tau) = \Psi(\tau^{-1})$ and note that Ψ_1 is summable on $[1,+\infty[$ iff $\Psi(\sigma)/\sigma^2$ is summable on $]0,1]$.

Now we define $\Psi_2(\tau) = \Psi_1(\lfloor\tau\rfloor)\cdot(\lceil\tau\rceil - \tau) + \Psi_1(\lceil\tau\rceil)\cdot(\tau - \lfloor\tau\rfloor)$. Using the Cauchy convergence criterion, it is easy to check that $\tau\Psi_2'(\tau)$ is summable iff Ψ_1 is summable. Also it is evident that $\Psi_2(\tau+2) < \Psi_1(\tau) < \Psi_2(\tau-2)$.

Finally, we mollify Ψ_2 so that $\widetilde{\Psi}_2(\tau+2) < \Psi_1(\tau) < \widetilde{\Psi}_2(\tau-2)$ and set $\Psi^\pm(\sigma) = \widetilde{\Psi}_2(\sigma^{-1} \mp 2)$, expanding Ψ^+ to $]\frac{1}{3}, 1]$ in a proper way. $\square$

In [S08, Theorem 1.8] the Hopf–Oleinik Lemma was proved for solution to (**NDE**) with $\mathbf{b} \equiv 0$ under assumption $0 \in \partial\Omega$ and $\Omega \supset Q_\Psi$, where

$$Q_\Psi = \{x \in \mathbb{R}^n : \ |x'| \le \sigma_0, \ \Psi(|x'|) < x_n < \sigma_0\},$$

while $\Psi(\sigma)/\sigma^2$ is summable.

By Lemma 2.4, this case can be reduced to $\Omega = Q_{\Psi^+}$. Then we again rectify the boundary and use part 2 of Theorem 2.3. In the same manner, [S08, Theorem 1.9] follows from part 1 of Theorem 2.3.

Remark 4. Note that the assumption (8) cannot be removed. Let us describe corresponding counterexample (see also [NU] and [S10]).

Let $u(x) = x_n \cdot \ln^\alpha(|x|^{-1})$ in $\mathcal{B}_{R,R}$. Then direct calculation shows that u satisfies an equation

$$-\Delta u + b_n(x) D_n u = 0 \qquad \text{with} \quad |b_n| \le \frac{C(\alpha)}{|x| \ln(|x|^{-1})} \in L_n(\mathcal{B}_{R,R}),$$

if R is small enough, and $u > 0 = u|_{x_n=0}$ in $\mathcal{B}_{R,R}$. However, it is easy to see that $D_n u(0) = 0$ for $\alpha < 0$ and $D_n u(0) = +\infty$ for $\alpha > 0$.

The condition (3) is also sharp. A simple one-dimensional counterexample is given in [A-Z]: the function $\phi(s) = \int\limits_0^s \exp\Big(- \int\limits_t^1 \frac{\omega(\tau)}{\tau}\, d\tau \Big) dt$ (cf. (9)) is positive on $]0,1]$, vanishes at zero and satisfies the equation

$$-\phi''(s) + \frac{\omega(s)}{s}\, \phi'(s) = 0.$$

However, if ω is not Dini continuous at zero then $\phi'(0) = 0$.

We describe also a more rich family of counterexamples generalizing [S08, Theorem 1.11].

Let Ω be a convex domain. Suppose that $\partial\Omega = \{x : \ x_n = \phi(x')\}$ at a neighborhood of the origin, $\phi \in \mathcal{C}^1$, $D'\phi(0) = 0$, and $\omega(\rho) = \sup\limits_{|x'| \le \rho} |D'\phi(x')|$ is not Dini continuous at zero. Let $\mathcal{L}$ be an operator of the form (**NDE**) in Ω with $\mathbf{b} = 0$, and let the condition (1) be satisfied.

It is shown in [AN11] that *any* solution of $\mathcal{L}u = 0$ positive in Ω and vanishing on $\partial\Omega$ at a neighborhood of the origin satisfies $\sup\limits_{|x|<\rho} \frac{u(x)}{\rho} \to 0$ as $\rho \to 0$.

Now we rectify $\partial\Omega$ at a neighborhood of the origin using the regularized distance and obtain a uniformly elliptic operator of the form (**NDE**) in $\mathcal{B}_{R,R}$ with $|\mathbf{b}(x)| \le \mathfrak{B}\, \frac{\omega(x_n/R)}{x_n}$ for which the Hopf–Oleinik lemma fails.

3 Parabolic case

In this section we assume $\partial_t u$, $D(Du) \in \widehat{L}_{q,\ell,loc}(Q)$ with some $q, \ell < \infty$ such that $\frac{n}{q} + \frac{1}{\ell} = 1$.

We recall an estimate which is a particular case of the statement in [N87, Sec.3]. For the isotropic case it was proved in [Kr86].

Proposition 3.1. *Let $\mathcal{M}$ be an operator of the form (**NDP**) in a cylinder $Q \subset B_R \times]0,T[$, and let the condition (1) be satisfied. Suppose that $\mathbf{b} \in L_\infty(Q)$, and a function $\mathbb{B}$ such that $\partial_t\mathbb{B}$, $D(D\mathbb{B}) \in L_\infty(Q)$ satisfies $\mathcal{M}\mathbb{B} \ge |\mathbf{b}|$ a.e. in Q. Then for any solution of $\mathcal{M}u = f$ in Q satisfying $u\big|_{\partial' Q} \le 0$, the following estimate holds:*

$$u \le N(n) \cdot \Big(\frac{\|\mathbb{B}\|_{\infty,Q} + R}{\nu} \Big)^{\frac{n}{q}} \cdot \| f_+ \|_{q,\ell,\{u>0\}}.$$

The next statement generalizes [AN95, Theorem 2]. For the isotropic case it was proved in [Li00, Theorem 5.2].

Lemma 3.1. *Let $\mathcal{M}$ be an operator of the form (*$\mathbf{NDP}$*) in a cylinder $\mathcal{Q}_{R,R}$, $R \le 1$, and let the condition (1) be satisfied. Suppose also that the vector function $\mathbf{b}$ can be written as follows:*

$$\mathbf{b} = \mathbf{b}^{(1)} + \mathbf{b}^{(2)}; \qquad |\mathbf{b}^{(1)}| \in \widehat{L}_{q,\ell}(\mathcal{Q}_{R,R}), \quad \frac{n}{q} + \frac{1}{\ell} = 1, \quad q, \ell < \infty, \tag{14}$$

$$|\mathbf{b}^{(2)}(x;t)| \le \mathfrak{B}\, \frac{\omega(x_n/R)}{x_n}, \qquad \omega \in \mathcal{D}_1. \tag{15}$$

Then for any solution of $\mathcal{M}u = f$ in $\mathcal{Q}_{R,R}$ satisfying $u\big|_{\partial'\mathcal{Q}_{R,R}} \le 0$, the following estimate holds:

$$u \le N_4 \cdot \left(\|\!|\mathbf{b}^{(1)}|\!\|^{\ell}_{q,\ell,\mathcal{Q}_{R,R}} + R \right)^{\frac{n}{q}} \cdot \|\!| f_+ |\!\|_{q,\ell,\{u>0\}}, \tag{16}$$

where N_4 depends only on n, ν, ℓ, $\mathfrak{B}$ and ω.

Proof. We consider a sequence of operators

$$\mathcal{M}_\varepsilon \equiv \partial_t - a_{ij\varepsilon}(x;t) D_i D_j + [b^{(1)}_{i\varepsilon}(x;t) + b^{(2)}_{i\varepsilon}(x;t)] D_i.$$

Here $a_{ij\varepsilon}$ are smooth functions satisfying (1) uniformly w.r.t. ε and tending to a_{ij} a.e. in $\mathcal{Q}_{R,R}$ as $\varepsilon \to 0$. Further,

$$b^{(1)}_{i\varepsilon}(x;t) = \max\{|b^{(1)}_i(x;t)|; \varepsilon^{-1}\} \cdot \mathrm{sign}(b^{(1)}_i(x;t));$$

$$b^{(2)}_{i\varepsilon}(x;t) = \begin{cases} b^{(2)}_i(x;t), & x_n > \varepsilon; \\ b^{(2)}_i(x', \varepsilon; t), & x_n \le \varepsilon. \end{cases}$$

Now we consider the boundary value problem

$$\mathcal{M}_\varepsilon v = \left(\frac{2\mathfrak{B}}{\nu}\, \mathcal{I}_\omega(1) + 1 \right) |\mathbf{b}^{(1)}_\varepsilon| + \mathfrak{B}\, \frac{\omega(\rho/R)}{\rho} \cdot \left(\frac{2\mathfrak{B}}{\nu}\, \mathcal{I}_\omega(1) - 1 \right)_+ \quad \text{in} \quad \mathcal{Q}_{R,R}; \qquad v\big|_{\partial'\mathcal{Q}_{R,R}} = 0,$$

where $\rho \le R$ will be chosen later. Denote by $\mathbb{B}^{(1)}_\varepsilon$ a unique solution of this BVP. By the maximum principle ([Kr76]), $\mathbb{B}^{(1)}_\varepsilon \ge 0$. Define

$$\mathbb{B}_\varepsilon(x;t) = \mathbb{B}^{(1)}_\varepsilon(x;t) + \frac{2\mathfrak{B}}{\nu}\, R \int\limits_{x_n/R}^{1} \mathcal{I}_\omega(s)\, ds.$$

Then

$$\mathcal{M}_\varepsilon \mathbb{B}_\varepsilon = \mathcal{M}_\varepsilon \mathbb{B}^{(1)}_\varepsilon + 2\mathfrak{B} \frac{a_{nn\varepsilon}}{\nu}\, \frac{\omega(x_n/R)}{x_n} - [b^{(1)}_{n\varepsilon} + b^{(2)}_{n\varepsilon}] \cdot \frac{2\mathfrak{B}}{\nu}\, \mathcal{I}_\omega(x_n/R) \ge |\mathbf{b}^{(1)}_\varepsilon| + |\mathbf{b}^{(2)}_\varepsilon| + F(x),$$

where

$$F(x) = \mathfrak{B}\, \frac{\omega(x_n/R)}{x_n} \cdot \left(1 - \frac{2\mathfrak{B}}{\nu}\, \mathcal{I}_\omega(x_n/R) \right) + \mathfrak{B}\, \frac{\omega(\rho/R)}{\rho} \cdot \left(\frac{2\mathfrak{B}}{\nu}\, \mathcal{I}_\omega(1) - 1 \right)_+.$$

We set $\rho = \min\{1; \widehat{s}\} R$, where $\widehat{s}$ is the root of $\mathcal{I}_\omega(s) = \frac{\nu}{2\mathfrak{B}}$. Then, for $0 < x_n < \rho$, we have

$$F(x) \ge \mathfrak{B}\, \frac{\omega(x_n/R)}{x_n} \cdot \left(1 - \frac{2\mathfrak{B}}{\nu}\, \mathcal{I}_\omega(\rho/R) \right) \ge 0.$$

If, otherwise, $\rho \le x_n \le R$, then

$$F(x) \ge \mathfrak{B}\,\frac{\omega(\rho/R)}{\rho}\cdot\Big(1-\frac{2\mathfrak{B}}{\nu}\,\mathcal{I}_\omega(1)\Big)+\mathfrak{B}\,\frac{\omega(\rho/R)}{\rho}\cdot\Big(\frac{2\mathfrak{B}}{\nu}\,\mathcal{I}_\omega(1)-1\Big)_+\ge 0.$$

So, in any case $\mathcal{M}_\varepsilon\mathbb{B}_\varepsilon \ge |\mathbf{b}_\varepsilon^{(1)}|+|\mathbf{b}_\varepsilon^{(2)}|$.

Using Proposition 3.1, we obtain the estimate

$$u \le N(n)\cdot\Big(\frac{\|\mathbb{B}_\varepsilon\|_{\infty,Q}+R}{\nu}\Big)^{\frac{n}{q}}\cdot|\!|\!|(\mathcal{M}_\varepsilon u)_+|\!|\!|_{q,\ell,\{u>0\}} \tag{17}$$

for any function u satisfying the assumptions of Lemma.

Obviously,

$$\|\mathbb{B}_\varepsilon\|_{\infty,Q}\le\|\mathbb{B}_\varepsilon^{(1)}\|_{\infty,Q}+\frac{2\mathfrak{B}}{\nu}\,R\mathcal{I}_\omega(1). \tag{18}$$

Further, the function $\mathbb{B}_\varepsilon^{(1)}$ itself satisfies the assumptions of Lemma. Therefore, one can set $u=\mathbb{B}_\varepsilon^{(1)}$ in (17) and use (18) arriving at

$$\|\mathbb{B}_\varepsilon^{(1)}\|_{\infty,Q}\le N(n)\cdot\Big(\frac{\|\mathbb{B}_\varepsilon^{(1)}\|_{\infty,Q}+R(1+\frac{2\mathfrak{B}}{\nu}\,\mathcal{I}_\omega(1))}{\nu}\Big)^{\frac{n}{q}}\cdot|\!|\!|(\mathcal{M}_\varepsilon\mathbb{B}_\varepsilon^{(1)})_+|\!|\!|_{q,\ell,\mathcal{Q}_{R,R}} \tag{19}$$

(we recall that $\mathbb{B}_\varepsilon^{(1)}\ge 0$).

If $\|\mathbb{B}_\varepsilon^{(1)}\|_{\infty,Q}>R(1+\frac{2\mathfrak{B}}{\nu}\,\mathcal{I}_\omega(1))$ then (19) gives

$$\frac{\|\mathbb{B}_\varepsilon^{(1)}\|_{\infty,Q}+R(1+\frac{2\mathfrak{B}}{\nu}\,\mathcal{I}_\omega(1))}{\nu}\le\Big(\frac{2N(n)}{\nu}\cdot|\!|\!|(\mathcal{M}_\varepsilon\mathbb{B}_\varepsilon^{(1)})_+|\!|\!|_{q,\ell,\mathcal{Q}_{R,R}}\Big)^{\ell}$$

(here we used $\frac{n}{q}+\frac{1}{\ell}=1$). Thus, in any case we have

$$\frac{\|\mathbb{B}_\varepsilon^{(1)}\|_{\infty,Q}+R(1+\frac{2\mathfrak{B}}{\nu}\,\mathcal{I}_\omega(1))}{\nu}\le\Big(\frac{2N(n)}{\nu}\cdot|\!|\!|(\mathcal{M}_\varepsilon\mathbb{B}_\varepsilon^{(1)})_+|\!|\!|_{q,\ell,\mathcal{Q}_{R,R}}\Big)^{\ell}+\frac{2R(1+\frac{2\mathfrak{B}}{\nu}\,\mathcal{I}_\omega(1))}{\nu}.$$

Substituting this estimate into (17) and taking into account the definition of $\mathbb{B}_\varepsilon^{(1)}$ we obtain (16) for $\mathcal{M}_\varepsilon$ instead of $\mathcal{M}$. Passage to the limit as $\varepsilon\to 0$ completes the proof. $\square$

The next Lemma is parabolic analog of Lemmas 2.2 and 2.2'.

Lemma 3.2. *Let $\mathcal{M}$ be an operator of the form (**NDP**) in $\mathcal{Q}_{\rho,\rho}$, $\rho\le R$, and let the conditions (1), (14) and (15) be satisfied. Suppose in addition that*

$$|\!|\!|\mathbf{b}^{(1)}|\!|\!|_{q,\ell,\mathcal{Q}_{\rho,\rho}}\le\mathfrak{A}\rho^{\frac{1}{\ell}}. \tag{20}$$

Let u be a nonnegative solution of $\mathcal{M}u=f\ge 0$ in $\mathcal{Q}_{\rho,\rho}$.

1. *If $u\ge k$ on $\partial'\mathcal{Q}_{\rho,\rho}\cap\{x_n=0\}$ for some $k>0$ then for $\xi\le\frac{1}{2}$ the inequality*

$$u\ge k\cdot\Big(\widehat{\beta}(n,\nu,\ell,\mathfrak{B},\mathfrak{A},\omega,\xi)-N_5(n,\nu,\ell,\mathfrak{B},\mathfrak{A},\omega,\xi)\cdot\big(\rho^{-\frac{1}{\ell}}|\!|\!|\mathbf{b}^{(1)}|\!|\!|_{q,\ell,\mathcal{Q}_{\rho,\rho}}+\mathfrak{B}\omega(\rho/R)\big)\Big) \tag{21}$$

holds in $\mathcal{Q}_{(1-\xi)\rho,(1-\xi)\rho}$ with some positive constants $\widehat{\beta}$ and N_5.

2. *If $u\ge k$ on $\partial'\mathcal{Q}_{\rho,\rho}\cap\{x_n=\rho\}$ for some $k>0$ then for $\xi\le\frac{1}{2}$ the inequality (21) holds in $\mathcal{Q}_{(1-\xi)\rho,\rho}\setminus\mathcal{Q}_{(1-\xi)\rho,\xi\rho}$.*

Proof. We prove the first statement. The proof of the second one is more simple, and we omit it.

First, let $\xi = \frac{1}{2}$. Consider the barrier function $\widehat{w}(x;t) = w(x) + \frac{t}{\rho^2}$, where w is defined in (5) with a constant $A \geq \frac{\sqrt{n-1}}{\nu} + \frac{1}{4\sqrt{n-1}}$ to be determined later.

Similarly to Lemma 2.2, direct calculation shows that $\mathcal{M}\widehat{w} \leq C_9(A,\nu,\mathfrak{B})|\mathbf{b}^{(1)}|\rho^{-1}$ in $\mathcal{Q}_{\rho,\frac{\rho}{A}}$.

Further, $\widehat{w}(x;t) \leq 0$ on $\partial'\mathcal{Q}_{\rho,\frac{\rho}{A}} \setminus \{x_n = 0\}$. Finally, $\widehat{w}(x;t) \leq 1$ on $\partial'\mathcal{Q}_{\rho,\frac{\rho}{A}} \cap \{x_n = 0\}$. This gives $k\widehat{w} - u \leq 0$ on $\partial'\mathcal{Q}_{\rho,\frac{\rho}{A}}$.

Lemma 3.1, condition (20) and relation $\frac{n}{q} + \frac{1}{\ell} = 1$ give for $(x;t) \in \mathcal{Q}_{\rho,\frac{\rho}{A}}$

$$u(x;t) \geq k\widehat{w}(x;t) - kN_4 \cdot \left(\mathfrak{A}^\ell + 1\right)^{\frac{n}{q}} \rho^{\frac{n}{q}} \cdot |\!|\!|(\mathcal{M}\widehat{w})_+|\!|\!|_{q,\ell,\mathcal{Q}_{\rho,\frac{\rho}{A}}} \geq k\widehat{w}(x;t) - kC_{10}\rho^{-\frac{1}{\ell}}|\!|\!|\mathbf{b}^{(1)}|\!|\!|_{q,\ell,\mathcal{Q}_{\rho,\rho}},$$

where N_4 is the constant from Lemma 3.1 while C_{10} depends only on n, ν, ℓ, A, $\mathfrak{B}$ and $\mathfrak{A}$.

Similarly to Lemma 2.2, one can choose $A = A(n,\nu,\mathfrak{B},\omega) \geq \frac{\sqrt{n-1}}{\nu} + \frac{1}{4\sqrt{n-1}}$ so large that $2(1+A)\varphi(1/A) \leq \frac{1}{100}$. Then direct calculation gives

$$u \geq k \cdot \left(\frac{1}{20} - C_{11}(n,\nu,\ell,\mathfrak{B},\mathfrak{A},\omega)\rho^{-\frac{1}{\ell}}|\!|\!|\mathbf{b}^{(1)}|\!|\!|_{q,\ell,\mathcal{Q}_{\rho,\rho}}\right)_+ \equiv k_2 \qquad \text{in} \quad \mathcal{B}_{\frac{\rho}{2},\frac{\rho}{10A}} \times \,]-\rho^2/2, 0[. \tag{22}$$

Now we consider the set $\widehat{K}_\rho = \mathcal{Q}_{\frac{3\rho}{4},\rho} \setminus \mathcal{Q}_{\frac{3\rho}{4},\frac{\rho}{20A}}$. Note that coefficients $b_i^{(2)}$ are bounded on this set, and

$$|\!|\!|(\mathbf{b}^{(2)}|\!|\!|_{q,\ell,\widehat{K}_\rho} \leq C_{12}(n)\mathfrak{B}\rho^{\frac{1}{\ell}}\Big(\int\limits_{\frac{1}{20A}}^{1}\Big(\frac{\omega(s\rho/R)}{s}\Big)^q ds\Big)^{\frac{1}{q}} \leq C_{13}(n,\nu,\ell,\mathfrak{B},\omega)\mathfrak{B}\rho^{\frac{1}{\ell}}\omega(\rho/R). \tag{23}$$

We proceed as [LU85, Lemma 3.2] (where the isotropic case was considered) and obtain

$$u \geq k_2 \cdot \Big(\widehat{\varkappa}(n,\nu,\ell,\mathfrak{B},\mathfrak{A},\omega) - C_{14}(n,\nu,\ell,\mathfrak{B},\mathfrak{A},\omega)\rho^{-\frac{1}{\ell}}|\!|\!|\mathbf{b}^{(1)} + \mathbf{b}^{(2)}|\!|\!|_{q,\ell,\widehat{K}}\Big) \qquad \text{in} \quad \mathcal{Q}_{\frac{\rho}{2},\frac{\rho}{2}} \setminus \mathcal{Q}_{\frac{\rho}{2},\frac{\rho}{10A}}.$$

By (22) and (23) the statement for $\xi = \frac{1}{2}$ follows.

For arbitrary $\xi < \frac{1}{2}$ we apply the obtained statement in cylinders $\mathcal{Q}_{2\xi\rho,2\xi\rho}(x^{0\prime};t^0)$ with $|x^{0\prime}| \leq (1-2\xi)\rho$, $(1-4\xi^2)\rho^2 \leq t^0 \leq 0$. We arrive at

$$u \geq k \cdot \Big(\frac{\widehat{\varkappa}}{20} - C_{15}(n,\nu,\ell,\mathfrak{B},\mathfrak{A},\omega)\big(\rho^{-\frac{1}{\ell}}|\!|\!|\mathbf{b}^{(1)}|\!|\!|_{q,\ell,\mathcal{Q}_{\rho,\rho}} + \mathfrak{B}\omega(\rho/R)\big)\Big)_+ \equiv k_3$$

in $\mathcal{B}_{(1-\xi)\rho,\xi\rho} \times \,]-(1-3\xi^2)\rho^2, 0[$.

Finally, as in the first step, one can proceed as [LU85, Lemma 3.2] in the set $\mathcal{Q}_{\rho,\rho} \setminus \mathcal{Q}_{\rho,\frac{\xi\rho}{2}}$, and (21) follows. $\square$

Remark 5. If we replace the assumption $f \geq 0$ by $f_- \in L_{q,\ell}(\mathcal{Q}_{\rho,\rho})$, the estimate (21) holds true with additional term $-N_6(n,\nu,\ell,\mathfrak{B},\mathfrak{A},\omega,\xi)\rho^{\frac{n}{q}} \cdot |\!|\!|f_-|\!|\!|_{q,\ell,\mathcal{Q}_{\rho,\rho}}$ in the right-hand side. The proof runs without changes.

Theorem 3.3. *Let $\mathcal{M}$ be an operator of the form (***NDP***) in $\mathcal{Q}_{R,R}$, and let the condition (1), (14) and (15) be satisfied. Suppose also that*

$$\mathfrak{A}_1(\rho) \equiv \sup_{Q_\rho(x^0;t^0) \subset \mathcal{Q}_{R,R}} \rho^{-\frac{1}{\ell}}|\!|\!|\mathbf{b}^{(1)}|\!|\!|_{q,\ell,Q_\rho(x^0;t^0)} \to 0, \qquad \rho \to 0, \tag{24}$$

and for $\rho \le R$

$$\sup_{\mathcal{Q}_{\rho,\rho}(x^{0\prime};t^0)\subset\mathcal{Q}_{R,R}} \rho^{-\frac{1}{\ell}} \|b_n^{(1)}\|_{q,\ell,\mathcal{Q}_{\rho,\rho}(x^{0\prime};t^0)} \le \mathfrak{B}_1\omega(\rho/R). \tag{25}$$

Then

1. *Any solution of $\mathcal{M}u = f \le 0$ in $\mathcal{Q}_{R,R}$ such that $u|_{x_n=0} \le 0$ and $u(0;0) = 0$ satisfies*

$$\sup_{0<x_n<R/2} \frac{u(0,x_n;0)}{x_n} \le \frac{N_7^+}{R} \cdot \sup_{\mathcal{Q}_{R/2,R/2}} u.$$

Consequently, if $D_n u(0;0)$ exists then $(D_n u)_+(0;0)$ is finite.

2. *Any positive solution of $\mathcal{M}u = f \ge 0$ in $\mathcal{Q}_{R,R}$ such that $u(0;0) = 0$ satisfies $\inf\limits_{0<x_n<R/2} \frac{u(0,x_n;0)}{x_n} > 0$. Consequently, if $D_n u(0;0)$ exists, it is positive. If, in addition, $f \equiv 0$, the following estimate holds:*

$$\inf_{0<x_n<R/2} \frac{u(0,x_n;0)}{x_n} \ge N_7^- \cdot \frac{u(0,R/2;-R^2/2)}{R/2}.$$

The constants $N_7^\pm$ depend on n, ν, ℓ, $\mathfrak{B}$, $\mathfrak{B}_1$, $\mathfrak{A}_1$ and ω.

Remark 6. If $|\mathbf{b}^{(1)}| \in \widehat{L}_{q,\widetilde{\ell}}(\mathcal{Q}_{R,R})$ such that $q, \widetilde{\ell} < \infty$ and $\frac{n}{q} + \frac{2}{\widetilde{\ell}} = 1$ then (24) is obviously satisfied.

Proof. Similarly to Theorem 2.3, we introduce the sequence of cylinders $\mathcal{Q}_{\rho_k,h_k}$, $k \ge 0$, where $\rho_k = 2^{-k}\rho_0$, $h_k = \zeta_k\rho_k$, while $\rho_0 \le R$ and the sequence $\zeta_k \downarrow 0$ will be chosen later.

Denote by $\widehat{M}_k^\pm$, $k \ge 1$, the quantities

$$\widehat{M}_k^+ = \sup_{\mathcal{Q}_{\rho_k,h_{k-1}}} \frac{u(x;t)}{\max\{x_n,h_k\}} \ge \sup_{\mathcal{Q}_{\rho_k,h_{k-1}}\setminus\mathcal{Q}_{\rho_k,h_k}} \frac{u(x;t)}{x_n}; \qquad \widehat{M}_k^- = \inf_{\mathcal{Q}_{\rho_k,h_{k-1}}\setminus\mathcal{Q}_{\rho_k,h_k}} \frac{u(x;t)}{x_n}.$$

Note that in the case **2** $\widehat{M}_k^- > 0$.

We define two function sequences

$$\widehat{v}_k^1 = u - \widehat{M}_k^+ h_k \frac{\varphi^+(x_n/R)}{\varphi^+(h_k/R)}; \qquad \widehat{v}_k^2 = \widehat{M}_k^- h_k \frac{\varphi^-(x_n/R)}{\varphi^-(h_k/R)} - u,$$

where functions $\varphi^\pm$ are introduced in (9), and denote

$$\begin{array}{llll} \widehat{V}_k = \widehat{v}_k^1; & \widehat{M}_k = \widehat{M}_k^+; & \Phi = \varphi^+ & \text{in the case } \mathbf{1}; \\ \widehat{V}_k = \widehat{v}_k^2; & \widehat{M}_k = \widehat{M}_k^-; & \Phi = \varphi^- & \text{in the case } \mathbf{2}. \end{array}$$

It is easy to see that $\widehat{V}_k\big|_{x_n=0} \le 0$ while the definition of $\widehat{M}_k$ gives $\widehat{V}_k\big|_{x_n=h_k} \le 0$.

To estimate $\widehat{V}_k$, we consider $(x^0;t^0) \in \mathcal{Q}_{\rho_k-h_k,h_k}$. Let $x_n^0 \le \frac{h_k}{2}$. Then we apply the first part of Lemma 3.2 to the function $\widehat{M}_k h_k - \widehat{V}_k$ in $\mathcal{Q}_{h_k,h_k}(x^{0\prime};t^0)$ (with regard to Remark 5). This gives for $x \in \mathcal{Q}_{\frac{h_k}{2},\frac{h_k}{2}}(x^{0\prime};t^0)$

$$\begin{aligned} \widehat{M}_k h_k - \widehat{V}_k(x) \ge \widehat{M}_k h_k \cdot \big[&\widehat{\beta}(n,\nu,\ell,\mathfrak{B},\mathfrak{A}_1(h_k),\omega,1/2) - \\ &- N_5(n,\nu,\ell,\mathfrak{B},\mathfrak{A}_1(h_k),\omega,1/2) \cdot (\mathfrak{A}_1(h_k) + \mathfrak{B}\omega(\rho_k/R))\big] - \\ &- N_6(n,\nu,\ell,\mathfrak{B},\mathfrak{A}_1(h_k),\omega,1/2) h_k^{\frac{n}{q}} \cdot \|(\mathcal{M}\widehat{V}_k)_+\|_{q,\ell,\mathcal{Q}_{h_k,h_k}(x^{0\prime};t^0)}. \end{aligned}$$

By (24), we can choose ρ_0/R is so small that the quantity in the square brackets is greater that $\frac{\widehat{\beta}}{2}$. As in Theorem 2.3, for $(x;t) \in \mathcal{Q}_{\frac{h_k}{2},\frac{h_k}{2}}(x^{0\prime};t^0)$ we arrive at

$$\widehat{V}_k(x;t) \le \widehat{M}_k h_k \cdot \Big[1 - \widehat{\beta}/2 + C_{16}(n,\nu,\ell,\mathfrak{B},\mathfrak{A}_1(h_k),\omega) h_k^{-\frac{1}{\ell}} \| b_n^{(1)} \|_{q,\ell,\mathcal{Q}_{h_k,h_k}(x^{0\prime};t^0)}\Big].$$

In particular, this estimate is valid for $(x;t) = (x^0;t^0)$. If $x_n^0 \ge \frac{h_k}{2}$, we get the same estimate using the second part of Lemma 3.2.

Taking supremum w.r.t $(x^0;t^0)$, we obtain

$$\sup_{\mathcal{Q}_{\rho_k - h_k, h_k}} \widehat{V}_k \le \widehat{M}_k h_k \cdot \big(1 - \widehat{\beta}/2 + C_{16}\mathfrak{B}_1 \omega(h_k/R)\big).$$

Repeating previous arguments provides for $m \le \frac{\rho_k}{h_k}$

$$\sup_{\mathcal{Q}_{\rho_k - m h_k, h_k}} \widehat{V}_k \le \widehat{M}_k h_k \cdot \Big((1 - \widehat{\beta}/2)^m + C_{16}\mathfrak{B}_1 \frac{\omega(h_k/R)}{\widehat{\beta}/2}\Big).$$

Setting $m = \lfloor \frac{\rho_{k+1}}{h_k} \rfloor$, we arrive at

$$\sup_{\mathcal{Q}_{\rho_{k+1}, h_k}} \widehat{V}_k \le \frac{\widehat{M}_k h_k}{1 - \widehat{\beta}/2} \cdot \Big(\exp\Big(-\widehat{\lambda}\frac{\rho_{k+1}}{h_k}\Big) + C_{16}\mathfrak{B}_1 \frac{\omega(h_k/R)}{\widehat{\beta}/2}\Big),$$

where $\widehat{\lambda} = -\ln(1 - \widehat{\beta}/2) > 0$.

Therefore, for $(x;t) \in \mathcal{Q}_{\rho_{k+1}, h_k}$

$$\frac{\widehat{V}_k(x;t)}{\max\{x_n, h_{k+1}\}} \le \widehat{M}_k \widehat{\gamma}_k, \tag{26}$$

where $\widehat{\gamma}_k = \frac{1}{1-\widehat{\beta}/2} \frac{\zeta_k}{2\zeta_{k+1}} \cdot \big(\exp\big(-\frac{\widehat{\lambda}}{2\zeta_k}\big) + C_{16}\mathfrak{B}_1 \frac{\omega(h_k/R)}{\widehat{\beta}/2}\big)$.

Estimate (26) implies in the cases **1** and **2**, respectively,

$$\widehat{M}_{k+1}^{\pm} \lesseqgtr \widehat{M}_k^{\pm}(\delta_k^{\pm} \pm \widehat{\gamma}_k),$$

where $\delta_k^{\pm}$ are defined in (13). Similarly to Theorem 2.3, we obtain

$$\widehat{M}_{k+1}^{\pm} \lesseqgtr \frac{\widehat{M}_1^{\pm}}{\Phi(1)} \cdot \prod_{j=1}^{k} (1 \pm \widehat{\gamma}_j \cdot \max\{1, \Phi(1)\}).$$

We set $\zeta_k = \frac{1}{k+k_0}$ and choose k_0 so large and ρ_0/R so small that $\widehat{\gamma}_1 \cdot \max\{1,\Phi(1)\} \le \frac{1}{2}$. Note that k_0 and ρ_0/R satisfying all the conditions imposed depend only on n, ν, $\mathfrak{B}$, $\mathfrak{B}_1$, $\mathfrak{A}_1$ and ω.

Now, as in Theorem 2.3, we observe that the series $\sum_k \widehat{\gamma}_k$ converges. Therefore, the infinite products $\widehat{\Pi}^{\pm} = \prod\limits_k (1 \pm \widehat{\gamma}_k \cdot \max\{1,\Phi(1)\})$ also converge, and we obtain in the cases **1** and **2**, respectively,

$$\widehat{M}_k^{\pm} \lesseqgtr \frac{\widehat{\Pi}^{\pm}\widehat{M}_1^{\pm}}{\Phi(1)}, \qquad k > 1.$$

Thus, all $\widehat{M}_k^{+}$ are bounded in the case **1**, and all $\widehat{M}_k^{-}$ are separated from zero in the case **2**.

Further, we note that $\widehat{M}_1^{+} \le \frac{1}{h_1} \sup\limits_{\mathcal{Q}_{R/2,R/2}} u$. This completes the proof of the statement **1**.

If $f \equiv 0$ then we set $\widehat{K} = \mathcal{Q}_{R,R} \setminus \mathcal{Q}_{R,h_1/2}$. Similarly to Lemma 3.2,

$$\sup_{Q_\rho(x^0,t^0)\subset\widehat{K}} \rho^{-\frac{1}{\ell}} |\!|\!| \mathbf{b}^{(1)} + \mathbf{b}^{(2)} |\!|\!|_{q,\ell,Q_\rho(x^0,t^0)} \to 0, \qquad \rho \to 0.$$

Therefore, we use the Harnack inequality which can be proved in a similar way as [S10, Theorem 3.3] (for bounded lower-order terms see [KrS]) and obtain

$$M_1^- \geq \frac{C_{17}}{h_1} \cdot u(0, R/2; -R^2/2),$$

where C_{17} depends on n, ν, ℓ, $\mathfrak{B}$, $\mathfrak{A}_1$ and ω. This completes the proof of the statement **2**. $\square$

Remark 7. If we replace in the case **1** the assumption $f \leq 0$ by $f = f^{(1)} + f^{(2)}$ with

$$\sup_{\mathcal{Q}_{\rho,\rho}(x^{0\prime};t^0)\subset\mathcal{Q}_{R,R}} \rho^{-\frac{1}{\ell}} |\!|\!| f_+^{(1)} |\!|\!|_{q,\ell,\mathcal{Q}_{\rho,\rho}(x^{0\prime};t^0)} \leq \mathfrak{F}_1 \omega(\rho/R), \qquad f_+^{(2)} \leq \mathfrak{F}_2 \frac{\omega(x_n/R)}{x_n},$$

the estimate

$$\sup_{0<x_n<R/2} \frac{u(0,x_n)}{x_n} \leq N_7^+ \cdot \Big(\frac{1}{R} \sup_{\mathcal{Q}_{R/2,R/2}} u + \mathfrak{F}_1 + \mathfrak{F}_2 \Big)$$

remains valid. The proof runs with minor changes.

Let us compare Theorem 3.3 with results known earlier. As in elliptic case, the proof of Hopf–Oleinik Lemma for classical solutions to parabolic equations with $b_i \in L_\infty$ works also for strong solutions by the Aleksandrov–Krylov maximum principle ([Kr76]; see also [N05]).

In [KHi] the Hopf–Oleinik Lemma was proved for classical solutions of (**NDP**) in $\mathcal{C}^{1+\mathcal{D},\frac{1}{2}+\mathcal{D}}$ domains. Using the parabolic regularized distance ([Li85, Theorem 3.1]) one can locally rectify the boundary and reduce the result of [KHi] to a particular case $\mathbf{b}^{(1)} \equiv 0$ of Theorem 3.3.

The boundary gradient estimates for solutions to (**NDP**) were established in [LU88] provided $|\mathbf{b}| \in L_{q+2}$, $q > n$; the Hopf–Oleinik Lemma under condition $|\mathbf{b}| \in \widehat{L}_{q,\widetilde{\ell}}$, $\frac{n}{q} + \frac{2}{\widetilde{\ell}} < 1$, $q, \widetilde{\ell} < \infty$, was announced in [NU]. In [AN95] the first part of Theorem 3.3 was proved for composite coefficients with $\mathbf{b}^{(1)} \in L_{q+2}$, $q > n$, and $\omega(\sigma) = \sigma^\alpha$, $\alpha \in]0,1[$.

Remark 8. Note that the assumption (25) cannot be removed. Let us describe corresponding counterexample (see [NU]).

Let $u(x;t) = x_n \cdot \ln^\alpha((|x|^2 - t)^{-1})$ in $\mathcal{Q}_{R,R}$. Then direct calculation shows that u satisfies an equation

$$\partial_t u - \Delta u + b_n(x;t) D_n u = 0 \qquad \text{with} \quad |b_n| \leq \frac{C(\alpha)}{(|x|^2 - t)^{\frac{1}{2}} \ln((|x|^2 - t)^{-1})} \in \widehat{L}_{q,\widetilde{\ell}}(\mathcal{Q}_{R,R}),$$

for any $q, \widetilde{\ell} < \infty$ such that $\frac{n}{q} + \frac{2}{\widetilde{\ell}} = 1$, if R is small enough. By Remark 6, the assumption (24) is satisfied. Moreover, $u > 0 = u|_{x_n=0}$ in $\mathcal{Q}_{R,R}$. However, it is easy to see that $D_n u(0;0) = 0$ for $\alpha < 0$ and $D_n u(0;0) = +\infty$ for $\alpha > 0$.

The condition (15) is also sharp. Indeed, considering functions depending only on spatial variables we see that the counterexample at the end of Section 2 works also for the parabolic operator (**NDP**). A more rich family of counterexamples also can be extracted from [AN11].

References

[Al] A.D. Aleksandrov, *Uniqueness conditions and bounds for the solution of the Dirichlet problem*, Vestnik Leningrad. Univ. Ser. Mat. Meh. Astronom. **18** (1963), N3, 5–29 (Russian).

[A-Z] R. Alvarado, D. Brigham, V. Maz'ya, M. Mitrea, E. Ziadé, *On the regularity of domains satisfying a uniform hour-glass condition and a sharp version of the Hopf–Oleinik boundary point principle*, Probl. Mat. Anal., **57** (2011), 3–68 (Russian); English transl.: J. Math. Sci., **176** (2011), N3, 281–360.

[AN95] D.E. Apushkinskaya, A.I. Nazarov, *Boundary estimates for the first-order derivatives of a solution to a nondivergent parabolic equation with composite right-hand side and coefficients of lower-order derivatives*, Probl. Mat. Anal., **14** (1995), 3–27 (Russian); English transl.: J. Math. Sci., **77** (1995), N4, 3257–3276.

[AN11] D.E. Apushkinskaya, A.I. Nazarov, *A counterexample to the Hopf–Oleinik lemma*, in preparation.

[G] G. Giraud, *Problèmes de valeurs à la frontière relatifs à certaines donnés discontinues*, Bull. de la Soc. Math. de France, **61** (1933), 1–54.

[GT] D. Gilbarg, N. Trudinger, *Elliptic Partial Differential Equations of Second Order*, 2nd ed., Springer, Berlin etc. (1983).

[Hi] B.N. Himčenko, *On the behavior of solutions of elliptic equations near the boundary of a domain of type $A^{(1)}$*, DAN SSSR **193** (1970), 304–305 (Russian); English transl.: Sov. Math. Dokl. **11** (1970), 943–944.

[Ho] E. Hopf, *A remark on linear elliptic differential equations of second order*, Proc. AMS, **3** (1952), 791–793.

[K] L.I. Kamynin, *A theorem on the interior derivative for a second-order uniformly parabolic equation*, DAN SSSR **299** (1988), N2, 280–283 (Russian); English transl.: Sov. Math. Dokl. **37** (1988), N2, 373–376.

[KHi] L.I. Kamynin, B.N. Himčenko, *The analogues of the Giraud theorem for a second order parabolic equation*, Sibirsk. Mat. Zh. **14** (1973), N1, 86–110 (Russian); English transl.: Siberian Math. J. **14** (1973), 59–77.

[Kr76] N.V. Krylov, *Sequences of convex functions, and estimates of the maximum of the solution of a parabolic equation*, Sibirsk. Mat. Zh. **17** (1976), N2, 290–303 (Russian); English transl.: Siberian Math. J. **17** (1976), N2, 226–236.

[Kr86] N.V. Krylov, *On estimates for the maximum of solutions of a parabolic equation and estimates for distribution of a semimartingal*, Mat. Sb. **130(172)** (1986), N2(6), 207–221 (Russian); English transl.: Math. USSR – Sbornik, **58** (1987), N1, 207–221.

[KrS] N.V. Krylov, M.V. Safonov, *A property of the solutions of parabolic equations with measurable coefficients*, Izv. AN SSSR Ser. Mat. **44** (1980), N1, 161–175 (Russian); English transl.: Math. USSR – Izvestiya, **16** (1981), N1, 151–164

[LU85] O.A. Ladyzhenskaya, N.N. Uraltseva, *Estimates of the Hölder constant for functions satisfying a uniformly elliptic or a uniformly parabolic quasilinear inequality with unbounded coefficients*, ZNS LOMI, **147** (1985), 72–94 (Russian); English transl.: J. Sov. Math., **37** (1987), N1, 837–851.

[LU88] O.A. Ladyzhenskaya, N.N. Uraltseva, *Estimates on the boundary of a domain for the first derivatives of functions satisfying an elliptic or parabolic inequality*, Trudy MI AN SSSR, **179** (1988), 102–125 (Russian); English transl.: Proc. Steklov Inst. Math., **179** (1989), 109–135.

[Li85] G.M. Lieberman, *Regularized distance and its applications*, Pacific J. Math. **117** (1985), N2, 329–352.

[Li86] G.M. Lieberman, *The Dirichlet problem for quasilinear elliptic equations with continuously differentiable boundary data*, Commun. PDEs, **11** (1986), N2, 167–229.

[Li00] G.M. Lieberman, *The maximum principle for equations with composite coefficients*, Electron. J. Diff. Eqs, N38 (2000), 17 pp.

[Na] N.S. Nadirashvili, *On the question of the uniqueness of the solution of the second boundary value problem for second-order elliptic equations*, Mat. Sb. **122(164)** (1983), N3, 341–359 (Russian); English transl.: Math. USSR – Sbornik, **50** (1985), N2, 325–341.

[N87] A.I. Nazarov, *Interpolation of linear spaces and estimates for the maximum of a solution for parabolic equations*, Partial differential equations, Akad. Nauk SSSR Sibirsk. Otdel., Inst. Mat., Novosibirsk, 1987, 50–72 (Russian).

[N01] A.I. Nazarov, *Estimates for the maximum of solutions of elliptic and parabolic equations in terms of weighted norms of the right-hand side*, Alg. & Anal. **13** (2001), N2, 151–164 (Russian); English transl.: St.Petersburg Math. J. **13** (2002), N2, 269–279.

[N05] A.I. Nazarov, *The A.D. Aleksandrov maximum principle*, Contemp. Math. Appl. **29** (2005), 127–143 (Russian); English transl.: J. Math. Sci., **142** (2007), N3, 2154–2171.

[NU] A.I. Nazarov, N.N. Ural'tseva, *Qualitative properties of solutions to elliptic and parabolic equations with unbounded lower-order coefficients*, St.Petersburg Math. Soc. El. Prepr. Archive. N 2009-05. 6pp.

[Ni] L. Nirenberg, *A strong maximum principle for parabolic equations*, Comm. Pure Appl. Math. **6** (1953), 167–177.

[O] O.A. Oleinik, *On properties of solutions of certain boundary problems for equations of elliptic type*, Mat. Sb. (N.S.), **30** (1952), 695–702 (Russian).

[S08] M.V. Safonov, *Boundary estimates for positive solutions to second order elliptic equations*, preprint, http://arxiv.org/abs/0810.0522 (2008), 20pp.

[S10] M.V. Safonov, *Non-divergence elliptic equations of second order with unbounded drift*, AMS Transl., Ser. 2, **229** (2010), 211–232.

[Z] M.S. Zaremba, *Sur un problème mixte relatif à l'équation de Laplace*, Bull. Intern. de l'Acad. Sci. de Cracovie, Ser. A, Sci. Math. (1910), 313–344.